PROJECTILE MOTION

1. BASIC CONCEPT :

1.1 PROJECTILE

Any object that is given an initial velocity obliquely and that subsequently follows a path determined by the gravitational force acting on it, is called a **Projectile**. A projectile may be a football, a cricket ball, or any other object.

1.2 TRAJECTORY

The path followed by a particle (here projectile) during its motion is called its **Trajectory**.

NOTE : 1. We shall consider only trajectories that are of sufficiently short range so that the gravitational force can be considered constant in both magnitude and direction.

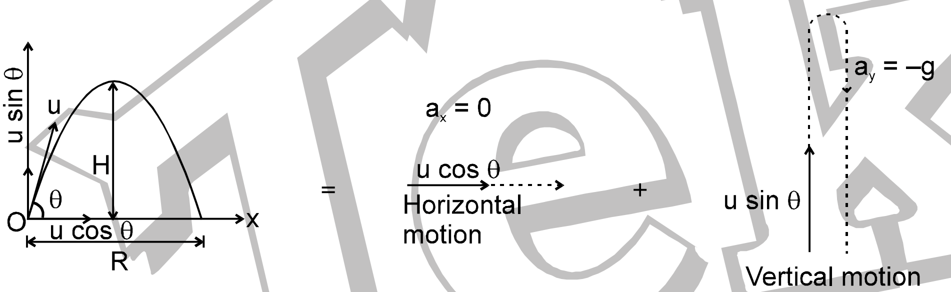
2. All effects of air resistance will be ignored; thus *our results are precise only for motion in a vacuum on flat, non rotating Earth.*

1.3 PROJECTILE MOTION

(i) The motion of projectile is known as projectile motion.

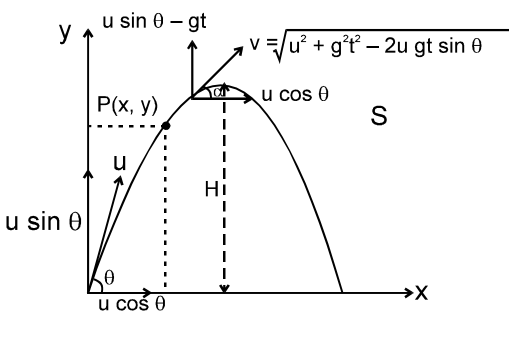
(ii) It is an example of two dimensional motion with constant acceleration.

(iii) Projectile motion is considered as combination of two simultaneous motions in mutually perpendicular directions which are completely independent from each other i.e. horizontal motion and vertical motion.



Parabolic path = vertical motion + horizontal motion.

2. PROJECTILE THROWN AT AN ANGLE WITH HORIZONTAL



(i) Consider a projectile thrown with a velocity u making an angle θ with the horizontal.

(ii) Initial velocity u is resolved in components in a coordinate system in which horizontal direction is taken as x-axis, vertical direction as y-axis and point of projection as origin.

$u_x = u \cos\theta$

$u_y = u \sin\theta$

(iii) Again this projectile motion can be considered as the combination of horizontal and vertical motion. Therefore,

	Horizontal direction	Vertical direction
(a)	Initial velocity $u_x = u \cos\theta$	Initial velocity $u_y = u \sin\theta$
(b)	Acceleration $a_x = 0$	Acceleration $a_y = g$
(c)	Velocity after time t, $v_x = u \cos\theta$	Velocity after time t, $v_y = u \sin\theta - gt$

2.1 TRAJECTORY EQUATION : If we consider the horizontal direction,

$x = u_x . t$

$x = u \cos\theta . t$ ...(1)

For vertical direction :

$y = u_y . t - 1/2 \, gt^2$

$= u \sin\theta . t - 1/2 \, gt^2$ ...(2)

Substituting the value x equation (1)

$$y = u \sin\theta . \frac{x}{u\cos\theta} - \frac{1}{2} g \left(\frac{x}{u\cos\theta}\right)^2$$

This is an equation of parabola called as **trajectory equation of projectile motion**.

2.2 TIME OF FLIGHT : Since the displacement along vertical direction does not occur. So,

Net displacement = 0

$(u \sin\theta) T - \frac{1}{2} gT^2 = 0$

$$T = \frac{2u\sin\theta}{g}$$

2.3 HORIZONTAL RANGE :

$R = u_x . T$

$R = u \cos\theta . \frac{2u\sin\theta}{g}$

$$R = \frac{u^2 \sin 2\theta}{g}$$

2.4 MAXIMUM HEIGHT :

Using 3rd equation of motion i.e.

$v^2 = u^2 + 2as$

we have for vertical direction

$0 = u^2 \sin^2\theta - 2gH$

$H = \frac{u^2 \sin^2\theta}{2g}$

2.5 RESULTANT VELOCITY :

$\vec{v} = v_x\hat{i} + v_y\hat{j}$

$= u\cos\theta\ \hat{i} + (u\sin\theta - gt)\ \hat{j}$

where $|\ \vec{v}\ | = \sqrt{u^2\cos^2\theta + (u\sin\theta - gt)^2}$ and $\tan\theta = v_y / v_x$.

NOTE:
- Results of article 2.2,2.4 and 2.4 are valid only for complete flight, that is when proejctile lands at same horizontal level from which it has been projected.
- Vertical component of velocity is zero when particle moves horizontally, i.e., at the highest point of trajectory.
- Vertical component of velocity is positive when particle is moving up and vertical component of velocity is negative when particle is coming down if vertical upwards direction is taken as positive. Any direction upward or downward can be taken as positive and if downward direction is taken as positive then vertical component of velocity coming down is positive.

2.6 GENERAL RESULT :

(i) For maximum range $\theta = 45^\circ$

$R_{max} = \frac{u^2}{g}$

In this situation

$H_{max} = \frac{R_{max}}{2}$

(ii) We get the same range for two angle of projections α and $(90 - \alpha)$ but in both cases, maximum heights attained by the particles are different.

(iii) If $R = H$

i.e. $\frac{u^2\sin 2\theta}{g} = \frac{u^2\sin^2\theta}{2g}$

$\Rightarrow$ $\tan\theta = 4$

(iv) Range can also be expressed as

$$R = \frac{u^2\sin 2\theta}{g} = \frac{2u\sin\theta . u\cos\theta}{g}$$

$$= \frac{2u_x u_y}{g}$$

(v) CHANGE IN MOMENTUM :

(a) Initial velocity $\vec{u}_i = u\cos\theta\ \hat{i} + u\sin\theta\ \hat{j}$

(b) Final velocity $\vec{u}_f = u\cos\theta\ \hat{i} - u\sin\theta\ \hat{j}$

Change in velocity for complete motion

$\Delta \vec{u} = \vec{u}_f - \vec{u}_i = -2u \sin\theta\ \hat{j}$

(c) Change in momentum for complete motion

$\Delta \vec{P} = \vec{P}_f - \vec{P}_i = m\ (\vec{u}_f - \vec{u}_i) = m(-2u \sin\theta)\ \hat{j}$

$= -2mu \sin\theta\ \hat{j}$

(d) Velocity at the highest point = $\vec{u}_f = u \cos\theta\ \hat{i}$

Change in momentum at highest point

$m\ (\vec{u}_f - \vec{u}_i) = m\ [u \cos\theta\ \hat{i} - (u \cos\theta\ \hat{i} + u \sin\theta\ \hat{j})\ ) = -mu \sin\theta\ \hat{j}$

Ex.1 A body is projected with a speed of 30 ms^{-1} at an angle of 30º with the vertical. Find the maximum height, time of flight and the horizontal range of the motion. [Take g = 10 m/s^2]

Sol. Here u = 30 ms^{-1}, Angle of projection, θ = 90 – 30 = 60º

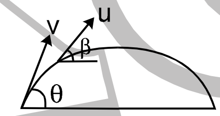
Maximum height, $H = \dfrac{u^2 \sin^2\theta}{2g} = \dfrac{30^2 \sin^2 60º}{10} = \dfrac{900}{20}\times\dfrac{3}{4} = \dfrac{135}{4}$ m

Time of flight, $T = \dfrac{2u\sin\theta}{g} = \dfrac{2\times30\times\sin60^0}{10} = 3\sqrt{3}$ sec.

Ex.2 A stone is thrown with a velocity v at angle θ with horizontal. Find its speed when it makes an angle β with the horizontal.

Sol. Since horizontal component of velocity remains constant. Therefore,

$v \cos\theta = u \cos\beta \qquad u = \dfrac{v\cos\theta}{\cos\beta}$



Ex.3 A projectile is thrown in the upward direction making an angle of 60^0 with the horizontal with a speed of 147 m/s. Find the time after which its inclination with the horizontal is 45^0 ?

Sol. $u_x = 147 \times \cos60^0 = \dfrac{147}{2}$

$u_y = 147 \times \sin60^0 = \dfrac{147\sqrt{3}}{2}$

$v_y = u_y + a_y t = \dfrac{147\sqrt{3}}{2} - gt$

$v_x = u_x = \dfrac{147}{2}$

When angle is 45^0, $\tan 45^0 = \dfrac{v_y}{v_x}$

$\Rightarrow \quad v_y = v_x \qquad \Rightarrow \quad \dfrac{147\sqrt{3}}{2} - gt = \dfrac{147}{2}$

$\Rightarrow \quad \frac{147}{2}(\sqrt{3}-1) = gt \qquad \Rightarrow \qquad t = \frac{147}{2\times10}(\sqrt{3}-1)\,s$

Ex.4 A large number of bullets are fired in all directions with the same speed v. What is the maximum area on the ground on which these bullets will spread ?

Sol. Maximum distance upto which a bullet can be fired is its maximum range, therefore

$$R_{max} = \frac{v^2}{g}$$

$$\text{Maximum area} = \pi(R_{max})^2 = \frac{\pi v^4}{g^2}.$$

Ex.5 The velocity of projection of a projectile is given by : $\vec{u} = 5\hat{i} + 10\hat{j}$. Find

(a) Time of flight, (b) Maximum height, (c) Range

Sol. We have $u_x = 5 \quad u_y = 10$

(a) $\text{Time of flight} = \frac{2u\sin\theta}{g} = \frac{2u_y}{g} = \frac{2\times10}{10} = 2\text{ s}$

(b) $\text{Maximum height} = \frac{u^2\sin^2\theta}{2g} = \frac{{u_y}^2}{2g} = \frac{10\times10}{2\times10} = 5\text{ m}$

(c) $\text{Range} = \frac{2u\sin\theta . u\cos\theta}{g} = \frac{2\times10\times5}{10} = 10\text{ m}$

HEIGHT AND RANGE :

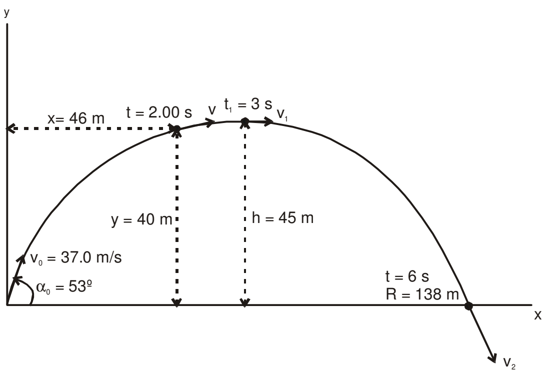
Ex.6 A batter hits a baseball so that it leaves the bat with an initial speed v_0 = 37.0 m/s at an initial angle α_0 = 53º, at a location where g = 10.0 m/s²

(a) Find the position of the ball, and the magnitude and direction of its velocity, when t = 2.0 s.

(b) Find the time when the ball, reaches the highest point of flight and the find its height h at this point

(c) Find the horizontal range R - that is, the horizontal distance from the starting point to the point at which the ball hits the ground.

For each part, treat the baseball as a projectile



Sol. The initial velocity of the ball has components

$v_{0x} = v_0 \cos\alpha_0 = (37.0\text{ m/s}) \cos 53º = 22.3\text{ m/s}$

$v_{0y} = v_0 \sin\alpha_0 = (37.0\text{ m/s}) \sin 53º = 29.5\text{ m/s}$

(a) $x = v_{0x}\,t = (22.3\text{ m/s})\,(2.00\text{ s}) = 44.6\text{ m}$

$y = v_{0y}\,t - \frac{1}{2}gt^2$

$= (29.5\text{ m/s})\,(2\text{ s}) - \frac{1}{2}(10\text{ m/s}^2)\,(2\text{ s})^2$

$= 59.0 - 20 = 39.0\text{ m}$

$v_x = v_{0x} = 22.3\text{ m/s}$

$v_y = v_{0y} - gt = 29.5$ m/s – (10 m/s^2) (2.00 s)

= 9.5 m/s

The y-component of velocity is positive, which means that the ball is still moving upward at this time (Figure). The magnitude and direction of the velocity are

$$v = \sqrt{v_x^2 + v_y^2} = \sqrt{(22.3\,m/s)^2 + (9.5\,m/s)^2} = 24.2 \text{ m/s}$$

$$\alpha = \tan^{-1}\left(\frac{10.0\,m/s}{22.3\,m/s}\right) = \tan^{-1} 0.4$$

(b) At the highest point, the vertical velocity v_y is zero at time t_1 ; then

$$v_y = 0 = v_{0y} - gt_1$$

$$t_1 = \frac{v_{0y}}{g} = \frac{29.5 \text{ m/s}}{10 \text{ m/s}^2} = 3.0 \text{ s}$$

The height h at this time is the value of y when $t = t_1 = 3$ s;

$$h = v_{0y} t_1 - \frac{1}{2} g t_1^2 = (29.5 \text{ m/s})(3.0 \text{ s}) - \frac{1}{2}(10.0 \text{ m/s}^2)(3.0 \text{ s})^2$$

= 88.5 – 45 = 43.5 m

(c) $$y = 0 = v_{0y} t_2 = \frac{1}{2} g t_2^2 = t_2 \left(v_{0y} - \frac{1}{2} g t_2\right)$$

This is a quadratic equation for t_2. It has two roots,

$$t_2 = 0 \quad \text{and} \quad t_2 = \frac{2v_{0y}}{g} = \frac{2(29.5\,m/s)}{10\,m/s^2} = 5.9 \text{ s}$$

There are two times at which y = 0; $t_2 = 0$ is the time the ball leaves the ground, and $t_2 = 6$ s is the time of its return. This is exactly twice the time to reach the highest point, so the time of descent equals the time of ascent. (This is always true if the starting and end points are at the same elevation and air resistance can be neglected).

The horizontal range R is the value of x when the ball returns to the ground, that is, at t = 6.64 s;

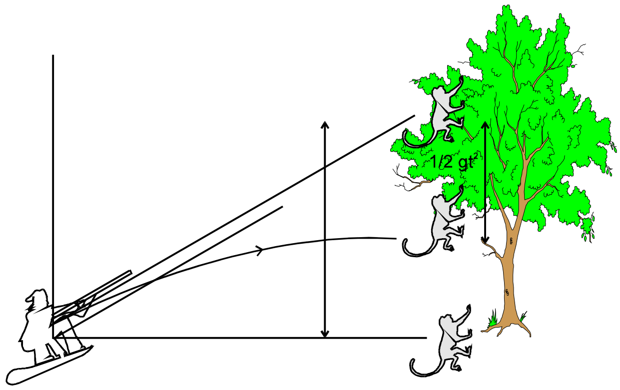
$$R = v_{0x} t_2 = (22.3 \text{ m/s})(5.9 \text{ s}) = 131.6 \text{ m}$$

The vertical component of velocity when the ball hits the ground is

$$v_y = v_{0y} - gt_2 = 29.5 \text{ m/s} - (10 \text{ m/s}^2)(5.9 \text{ s}) = -29.5 \text{ m}$$

That is, v_y has the same magnitude as the initial vertical velocity v_{0y} but the opposite direction (down). Since v_x is constant, the angle $\alpha = -53º$)(below the horizontal) at the point is the negative of the initial angle $\alpha_0 = 53º$.

Ex.7 A clever monkey escapes from the zoo. The zoo keeper finds him in a tree. After failing to entice the monkey and shoots. The clever monkey lets go at the same instant the dart leaves the gun barrel, intending to land on the ground and escape. Show that the dart always hits the monkey, regardless of the dart's muzzle velocity (provided that it gets to the monkey before he hits the ground).



Sol. The monkey drops straight down, so $x_{monkey} = d$ at all times. For the dart, $x_{dart} = (v_0 \cos a_0)t$. When these x-coordinates are equal, $d = (v_0 \cos \alpha_0)t$, or

$$t = \frac{d}{v_0 \cos \alpha_0}$$

To have the dart hit the monkey, it must be true that $y_{monkey} = y_{dart}$ at this same time. The monkey is in one-dimensional free fall $\quad y_{monkey} = d \tan \alpha_0 - \frac{1}{2} gt^2$.

For the dart $\quad y_{dart} = (v_0 \sin \alpha_0)\, t - \frac{1}{2}\, gt^2$

So if $d \tan \alpha_0 = (v_0 \sin \alpha_0)\, t$ at the time when the two x-coordinates are equal, then $y_{monkey} = y_{dart}$ and we have a hit.

Ex.8 A ball is thrwon from ground level so as to just clear a wall 4 m high at a distance of 4 m and falls at a distance of 14 m from the wall. Find the magnitude and direction of the ball.

Sol. The ball passes through the point P(4, 4). So its range = 4 + 14 = 18 m.

The trajectory of the the ball is,

$$y = x \tan \theta \left(1 - \frac{x}{R}\right)$$

Now $\quad x = 4m,\ y = 4m$ and $R = 18$ m

$\therefore \quad 4 = 4 \tan \theta \left[1 - \frac{4}{18}\right] = 4 \tan \theta \cdot \frac{7}{9}$

or $\quad \tan\theta = \frac{9}{7} \quad \Rightarrow \quad \theta = \tan^{-1}\frac{9}{7}$

And $\quad R = \frac{2u^2 \sin\theta \cos\theta}{g}$

or $\quad 18 = \frac{2}{9.8} \times u^2 \times \frac{9}{\sqrt{130}} \times \frac{7}{\sqrt{130}}$

or $\quad u^2 = \frac{18 \times 9.8 \times 130}{2 \times 9 \times 7} = 182$

or $\quad u = \sqrt{182} \qquad$ and $\theta = \tan^{-1}\frac{9}{7}$.

Qus. Two projectiles are thrown with different speeds and at different angles so as the cover the same maximum height. Find out the sum of the times taken by each to the reach to highest point, if time of flight is T.

Ans. Total time taken by either of the projectile.

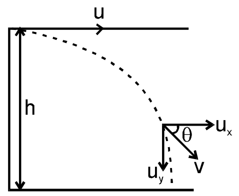
Qus. A particle is projected with speed 10 m/s at an angle 60^0 with horizontal. Find :
(a) time of flight (b) range
(c) maximum height (d) velocity of particle after one second.
(e) velocity when height of the particle is 1 m

Ans. (a) $\sqrt{3}$ sec. (b) $5\sqrt{3}$ m (c) $\frac{15}{4}$ m (d) 5.17 m/s

(e) $\vec{v} = 5\hat{i} \pm \sqrt{55}\,\hat{j}$

3. PROJECTILE THROWN PARALLEL TO THE HORIZONTAL :

Consider a projectile thrown from point O at some height h from the ground with a velocity u. Now we shall deal the characteristics of projectile motion with the help of horizontal and vertical direction motions.



	Horizontal direction	**Vertical direction**
(i)	Initial velocity $u_x = u$	Initial velocity $u_y = 0$
(ii)	Acceleration $a_x = 0$	Acceleration $a_y = g$ (downward)

3.1 TRAJECTORY EQUATION : The path traced by projectile is called the trajectory.

After time t,

$$x = ut \qquad ....(1)$$

$$y = \frac{-1}{2} gt^2 \qquad ....(2)$$

From equation (1) $t = x/u$

Put value of t in equation (2)

$$y = \frac{-1}{2} g \cdot \frac{x^2}{u^2}$$

This is **trajectory equation of the projectile.**

3.2 Velocity at a general point P(x, y)

$$v = \sqrt{u_x^2 + u_y^2}$$

Here horizontal velocity of the projectile after time t

$$v_x = u$$

velocity of projectile in vertical direction after time t

$$v_y = 0 + (-g)t = -gt = gt \text{ (downward)}$$

$\therefore \quad v = \sqrt{u^2 + g^2t^2}$ and $\tan\theta = v_y/v_x$

3.3 DISPLACEMENT : The displacement of the particle is expressed by

$$S = x\,\hat{i} + y\,\hat{j}$$

$$= (ut)\,\hat{i} + (\frac{1}{2}gt^2)\,\hat{j}$$

where $|S| = \sqrt{x^2 + y^2}$

3.4 TIME OF FLIGHT : This is equal to the time taken by the projectile to return to ground. From equation of motion

$$S = ut + \frac{1}{2}at^2$$

Therefore for vertical direction $\quad -h = v_y t + \frac{1}{2}(-g)t^2$

At highest point $v_y = 0 \quad \Rightarrow \quad h = \frac{1}{2}gt^2$

$t = \pm\sqrt{\frac{2h}{g}} \quad \Rightarrow \quad t = \sqrt{\frac{2h}{g}}$

3.5 HORIZONTAL RANGE : Distance covered by the projectile along the horizontal direction between the point of projection to the point on the ground.

$R = u_x \cdot t$

$$R = u\sqrt{\frac{2h}{g}}$$

3.6 VELOCITY AT VERTICAL DEPTH h :

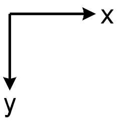
Along vertical direction $\quad v_y^2 = 0^2 + 2 \cdot (-h)(-g)$

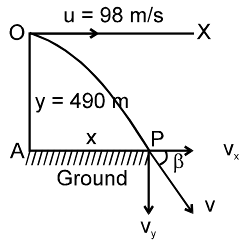
$$v_y = \sqrt{2gh}$$

EXAMPLES BASED ON PROJECTILE PROJECTED HORIZONTALLY

Ex.9 A projectile is fired horizontally with a speed of 98 ms^{-1} from the top of a hill 490 m high. Find (i) the time taken to reach the ground (ii) the distance of the target from the hill and (iii) the velocity with which the projectile hits the ground. (take g = 9.8 m/s^2)

Sol. (i) The projectile is fired from the top O of a hill with speed u = 98 ms^{-1} along the horizontal as shown as OX. It reaches the target P at vertical depth OA, in the coordinate system as shown, OA = y = 490 m





As, $y = \frac{1}{2} gt^2$

$\therefore$ $490 = \frac{1}{2} \times 9.8\, t^2$

or $t = \sqrt{100} = 10$ s.

(ii) Distance of the target from the hill is given by,

AP = x = Horizontal velocity × time = 98 × 10 = 980 m.

(iii) The horizontal and vertical components of velocity v of the projectile at point P are

$v_x = u = 98\ ms^{-1}$

$v_y = u_y + gt = 0 + 9.8 \times 10 = 98\ ms^{-1}$

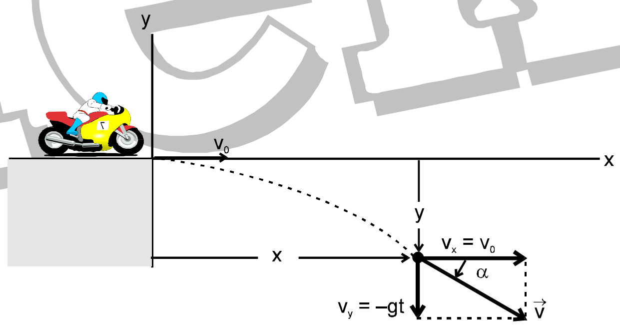
$V = \sqrt{v_x^2 + v_y^2} = \sqrt{98^2 + 98^2} = 98\sqrt{2} = 139\ ms^{-1}$

Now if the resultant velocity v makes air angle β with the horizontal, then

$\tan\beta = \frac{v_y}{v_u} = \frac{98}{98} = 1$ $\therefore$ $\beta = 45º$

Ex.10 A motorcycle stunt rider rides off the edge of a cliff. Just at the edge his velocity is horizontal, with magnitude 9.0 m/s. Find the motorcycle's position, distance from the edge of the cliff and velocity after 0.5 s.

Sol.



At t = 0.50 s, the x and y-coordinates are

$x = v_{0x}t = (9.0\ m/s)\ (0.50\ s) = 4.5\ m$

$y = -\frac{1}{2}\, gt^2 = -\frac{1}{2}\, (10\ m/s^2)\ (0.50\ s)^2 = -1.2\ m$

The negative value of y shows that a this time the motorcycle is below its starting point.

The motorcycle's distance from the origin at this time

$r = \sqrt{x^2 + y^2} = \sqrt{(4.5m)^2 + (-1.2m)^2} = \sqrt{\left(\frac{45}{10}m\right)^2 + \left(-\frac{12}{10}m\right)^2} = \frac{3}{10}\sqrt{(15)^2 + (4)^2} \simeq 5$ sec.

The components of velocity at this time are

$v_x = v_{0x} = 9.0$ m/s

$v_y = -gt = (-10\ m/s^2)(0.50\ s) = -5$ m/s.

The speed (magnitude of the velocity) at this time is

$$v = \sqrt{v_x^2 + v_y^2} \qquad = \sqrt{(9.0\,m/s)^2 + (-5\,m/s)^2} = 10.2\ m/s$$

The angle α of the velocity vector is

$$= \tan^{-1}\frac{v_y}{v_x} \qquad = \tan^{-1}\left(\frac{-5\ m/s}{9.0\ m/s}\right)$$

Qus. Two tall buildings face each other and are at a distance of 180 m from each other. With what velocity must a ball be thrown horizontally from a window 55 m above the ground in one building, so that it enters a window 10.9 m above the ground in the second building.

Ans. 60 m/s.

Qus. Two paper screens A and B are separated by a distance of 100 m. A bullet pierces A and then B. The hole in B is 10 cm below the hole in A. If the bullet is travelling horizontally at the time of hitting the screen A, calculate the velocity of the bullet when it hits the screen A. Neglect the resistance of paper and air.

Ans. 700 m/s

4. PROJECTILE FROM A TOWER

Case (i) : Horizontal projection

$u_x = u$; $u_y = 0$; $a_y = -g$

Case (ii) : Projection at an angle θ above horizontal

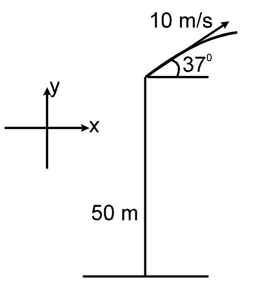
$u_x = u\cos\theta$; $u_y = u\sin\theta$; $a_y = -g$

Case (iii) : Projection at an angle θ below horizontal

$u_x = u\cos\theta$; $u_y = -u\sin\theta$; $a_y = -g$

In all the above three cases, we can calculate the velocity of projectile at the instant of striking the ground by using $v = \sqrt{v_x^2 + v_y^2}$ and $\tan\phi = \frac{v_y}{v_x}$, where ϕ is the angle at which the projectile strikes the ground.

Ex.11 From the top of a 50m high tower a stone is projected with speed 10 m/s, at an angle of 37º as shown in figure. Find out (a) velocity after 3s (b) time of flight. (c) horizontal range. (d) the maximum height attained by the particle.



Sol. (a) Initial velocity in horizontal direction = 10 cos 37 = 8 m/s
Initial velocity in vertical direction = 10 sin 37º = 6 m/s
Velocity after 3 seconds

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$v = v_x\,\hat{i} + v_y\,\hat{j}$

$= 8\,\hat{i} + (u_y + a_y t)\,\hat{j}$

$= 8\hat{i} - 24\,\hat{j}$

(b) $S_y = u_y t + \frac{1}{2} a_y t^2 \quad \Rightarrow \quad -50 = 6 \times t + \frac{1}{2} \times (-10)\, t^2$

$5t^2 - 6t - 50 = 0 \quad \Rightarrow \quad t = \frac{6 \pm \sqrt{1036}}{10}$

(c) Range $= 8 \times \left(\frac{6 \pm \sqrt{1036}}{10}\right)$

(d) $v_y = u_y + a_y t$
$0 = 6 - 10t$
$t = 0.6$
or $0 = 6 - 2 \times 10 \times h$
$h = 1.8$
maximum height = 50 + 1.8 = 51.8 m.

Qus. Two stones A and B are projected simultaneously from the top of a 100 m high tower. Stone B is projected horizontally with speed 10 m/s, and stone A is dropped from the tower. Find out the following
(a) time of flight of the two stone.
(b) distance between two stones after 3 sec.
(c) angle of strike with ground.
(d) horizontal range of particle B.

Ans. (a) $2\sqrt{5}$ sec. (b) $x_B = 30$ m, $y_B = 45$ (c) $\tan^{-1} 2\sqrt{5}$ (d) $20\sqrt{5}$ m

5. PROJECTION FROM A MOVING BODY

Consider a boy standing on a trolley who throws a ball with speed u at an angle θ with the horizontal. Trolley moves horizontally with constant speed v .

Case (i) :

When ball is projected in the direction of motion of the trolley, horizontal component of ball's velocity = u cos θ + v Initial vertical component of ball's velocity = u sin θ

Case (ii) : The ball is projected opposite to the direction of motion of the trolley

Horizontal component of ball's velocity = u cosθ − v

Initial vertical component of ball's velocity = u sinθ

Case (iii) : The ball projected upwards from a platform moving with speed v upwards.

Horizontal component of ball's velocity = u cos θ

Initial vertical component of ball's velocity = u sin θ + v

Case (iv) : The ball projected upwards from a platform moving with speed v downwards.

Horizontal component of ball's velocity = u cosθ

Initial vertical component of ball's velocity = u sinθ − v

Ex.12 A particle is projected at an angle of 30^0 with speed 20 m/s :

(i) Find out position vector of the particle after 1s
(ii) Find out angle betwen velocity vector and position vector at t = 1s

Sol. (i) $s_x = u \cos\theta\, t$

$= 20 \times \frac{\sqrt{3}}{2} \times t \qquad = 10\sqrt{3}$ m

$s_y = u \sin\theta t + \frac{1}{2} \times 10 \times t^2$

$= 20 \times \frac{1}{2} \times 1 - 5\,(1)^2 \quad = 5$m

Position vector = $10\sqrt{3}\,\hat{i} + 5\,\hat{j}$

(ii) $v_x = 10\sqrt{3}\,\hat{i}$

$v_y = u_y + a_y t = 10 - 10 = 0$

$\vec{v} = 10\sqrt{3}\,\hat{i}$ $\qquad \vec{v}\,.\,\vec{s} = |v|\,|s| \cos\theta$

$\cos\theta = \frac{10\sqrt{3} \times 10\sqrt{3}}{10\sqrt{3} \times \sqrt{325}} = \frac{10\sqrt{3}}{\sqrt{325}} = 2\sqrt{\frac{3}{13}}$

$\theta = \cos^{-1}\left(2\sqrt{\frac{3}{13}}\right)$

Ex.13 A boy standing on a long railroad car throws a ball straight upwards. The car is moving on the horizontal road with an acceleration of 1 m/s^2 and the projection speed in the vertical direction is 9.8 m/s. How far behind the boy will the ball fall on the car ?

Sol. Let the initial velocity of car be 'u'. $\qquad t = \frac{2u_\perp}{g} = 2$

where $u_\perp$ = component of velocity in vertical direction

$x_c = u \times 2 + \frac{1}{2} \times 1 \times 2^2$

$= 2u + 2$

where x_c = distance travelled by car

x_b = distance travelled by ball

$x_b = 2u$

$x_c - x_b = 2u + 2 - 2u$

$= 2$m **Ans.**

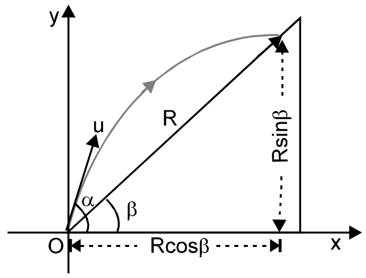
Qus. A person is standing on a truck moving with a constant velocity of 14.7 m/s on a horizontal road. The man throws a ball in such a way that it returns to the truck after the truck has moved 58.8m. Find the speed and the angle of projection (a) as seen from the truck , (b) as seen from the road.
[Ans : (a) 19.6 m/s upward (b) 24.5 m/s at 53º with horizontal]

6. PROJECTION ON AN INCLINED PLANE

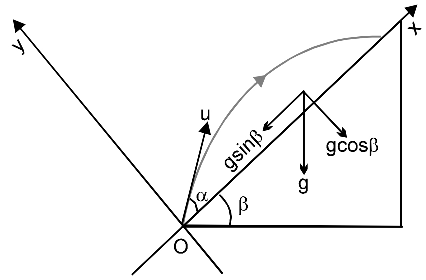
To solve the problem of projectile motion on an incline plane we can adopt two types of axis system as shown in the figures

Case (i) :

Up the incline



axis system 1



axis system 2

Here α is angle of projection with the horizontal.
In this case:

$a_x = 0$ $\quad u_x = u\cos\alpha$

$a_y = -g$ $\quad u_y = u\sin\alpha$

Here α is angle of projection with the inclined plane
In this case:

$a_x = -g\sin\beta$ $\quad u_x = u\cos\alpha$

$a_y = -g\cos\beta$ $\quad u_y = u\sin\alpha$

Time of flight (T) :

when the particle strikes the inclined plane y coordinate becomes zero

$$y = u_y t + \frac{1}{2}a_y t^2$$

$$\Rightarrow \quad 0 = u\sin\alpha T - \frac{1}{2}g\cos\beta T^2$$

$$\Rightarrow \quad T = \frac{2u\sin\alpha}{g\cos\beta} = \frac{2u_\perp}{g_\perp}$$

Maximum height (H) :
when half of the time is elasped y coordinate is equal to maximum height of the projectile

$$H = u\sin\alpha\left(\frac{u\sin\alpha}{g\cos\beta}\right) - \frac{1}{2}g\sin\beta\left(\frac{u\sin\alpha}{g\cos\beta}\right)^2$$

$$\Rightarrow \quad H = \frac{u^2\sin^2\alpha}{2g\cos\beta} = \frac{u^2{}_\perp}{2g_\perp}$$

Range along the inclined plane (R):
When the particle strikes the inclined plane x coordinate is equal to range of the particle

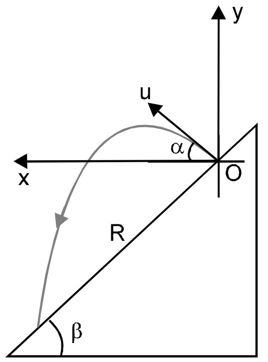
$$x = u_x t + \frac{1}{2}a_x t^2$$

$$\Rightarrow \quad R = u\cos\theta\left(\frac{2u\sin\alpha}{g\cos\beta}\right) - \frac{1}{2}g\cos\beta\left(\frac{2u\sin\alpha}{g\cos\beta}\right)^2$$

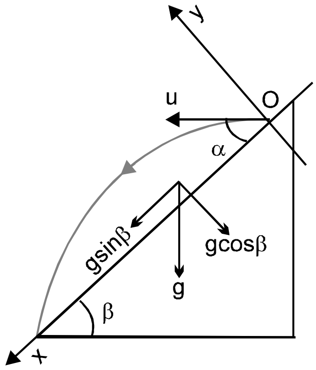
$$\Rightarrow \quad R = \frac{2u^2\sin\alpha\cos(\alpha+\beta)}{g\cos^2\beta}$$

Case (ii) :

Down the incline



axis system 1



axis system 2

In this case :

$a_x = 0$ $\quad u_x = u\cos\alpha$

$a_y = -g$ $\quad u_y = u\sin\alpha$

In this case :

$a_x = g\sin\beta$ $\quad u_x = u\cos\alpha$

$a_y = -g\cos\beta$ $\quad u_y = u\sin\alpha$

Time of flight (T) :

when the particle strikes the inclined plane y coordinate becomes zero

$$y = u_y t + \frac{1}{2} a_y t^2$$

$$\Rightarrow \quad 0 = u\sin\alpha T - \frac{1}{2} g\cos\beta T^2$$

$$\Rightarrow \quad T = \frac{2u\sin\alpha}{g\cos\beta} = \frac{2u_\perp}{g_\perp}$$

Maximum height (H) :

when half of the time is elasped y coordinate is equal to maximum height of the projectile

$$H = u\sin\alpha\left(\frac{u\sin\alpha}{g\cos\beta}\right) - \frac{1}{2} g\sin\beta\left(\frac{u\sin\alpha}{g\cos\beta}\right)^2$$

$$\Rightarrow \quad H = \frac{u^2\sin^2\alpha}{2g\cos\beta} = \frac{u^2{}_\perp}{2g_\perp}$$

Range along the inclined plane (R):

When the particle strikes the inclined plane x coordinate is equal to range of the particle

$$x = u_x t + \frac{1}{2} a_x t^2$$

$$\Rightarrow \quad R = u\cos\theta\left(\frac{2u\sin\alpha}{g\cos\beta}\right) + \frac{1}{2} g\cos\beta\left(\frac{2u\sin\alpha}{g\cos\beta}\right)^2$$

$$\Rightarrow \quad R = \frac{2u^2\sin\alpha\cos(\alpha-\beta)}{g\cos^2\beta}$$

Table 1 : Standard results for projectile motion on an inclined plane

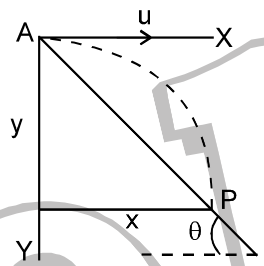
	Up the Incline	Down the Incline
Range	$\dfrac{2u^2\sin\alpha\cos(\alpha+\beta)}{g\cos^2\beta}$	$\dfrac{2u^2\sin\alpha\cos(\alpha-\beta)}{g\cos^2\beta}$
Time of flight	$\dfrac{2u\sin\alpha}{g\cos\beta}$	$\dfrac{2u\sin\alpha}{g\cos\beta}$
Angle of projection for maximum range	$\dfrac{\pi}{4}-\dfrac{\beta}{2}$	$\dfrac{\pi}{4}+\dfrac{\beta}{2}$
Maximum Range	$\dfrac{u^2}{g(1+\sin\beta)}$	$\dfrac{u^2}{g(1-\sin\beta)}$

Here α is the angle of projection with the incline and β is the angle of incline.

NOTE : For a given speed, the direction - which gives the maximum range of the projectile on an incline, bisects the angle between the incline and the vertical, for upward or downward projection.

Ex.14 A particle is projected horizontally with a speed u from the top of a plane inclined at an angle θ with the horizontal. How far from the point of projection will the particle strike the plane?

Sol. Take X,Y-axes as shown in figure. Suppose that the particle strikes the plane at a point P with coordinates (x,y) . Consider the motion between A and P.



Motion in x direction :

Initial velocity = u

Acceleration = 0

$x = ut$......................(i)

Motion in y direction :

Initial velocity = u

Acceleration = g

$$y = \frac{1}{2}gt^2 \text{.......................(ii)}$$

Eliminating t from (i) and (ii)

$$y = \frac{1}{2}g\frac{x^2}{u^2}$$

Also $y = x\tan\theta$

Thus , $\dfrac{gx^2}{gu^2} = x\tan\theta$ giving x = 0 or, $\dfrac{2u^2\tan\theta}{g}$

Clearly the point P corresponds to $x = \dfrac{2u^2\tan\theta}{g}$

then $y = x\tan\theta = \dfrac{2u^2\tan^2\theta}{g}$

The distance AP = $\sqrt{x^2+y^2}$

$$= \frac{2u^2}{g}\tan\theta\sqrt{1+\tan^2\theta} \qquad = \frac{2u^2}{g}\tan\theta\sec\theta$$

Ex.15 A projectile is thrown at an angle θ with an inclined plane of inclination β as shown in figure. Find the relation between β and θ if :

(a) projectile strikes the inclined plane perpendicularly,

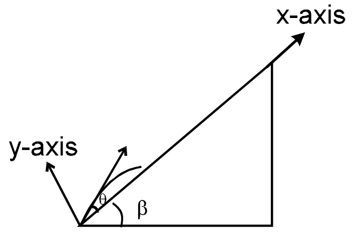
(b) projectile strikes the inclined plane horizontal.

Sol. (a) If projectile strikes perpendicularly.

$v_x = 0$ when projectile strikes

$v_x = u_x + a_x t$

$0 = u\cos\theta - g\sin\beta\, T$



$$T = \frac{u\cos\theta}{g\sin\beta}$$

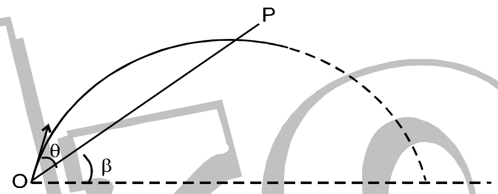
we also know that $T = \frac{2u\sin\theta}{g\cos\beta}$

$$\Rightarrow \quad \frac{u\cos\theta}{g\sin\beta} = \frac{2u\sin\theta}{g\cos\beta} \quad \Rightarrow \quad 2\tan\theta = \cot\beta$$

(b) If projectile strikes horizontally, then at the time of striking the projectile will be at the maximum height from the ground. Therefore :

$$t_{0P} = \frac{2u\sin\theta}{g\cos\beta}$$

$$t_{0P} = \frac{2u\sin(\theta+\beta)}{2\times g}$$



$$\Rightarrow \quad \frac{2u\sin\theta}{g\cos\beta} = \frac{2u\sin(\theta+\beta)}{2g}$$

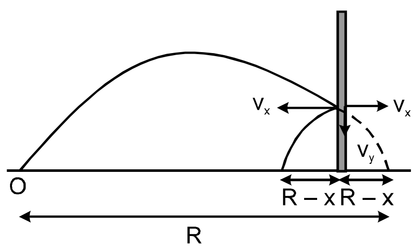
$$\Rightarrow \quad 2\sin\theta = \sin(\theta+\beta)\cos\beta\,.$$

Elastic collision of a projectile with a wall :

Suppose a projectile is projected with speed u at an angle θ from point O on the ground. Range of the projectile is R. If a wall is present in the path of the projectile at a distance x from the point O. The collision with the wall is elastic, path of the projectile changes after the collision as described below.

Case I : If $x \geq \frac{R}{2}$

Direction of x component of velocity is reversed but its magnitude remains the same and y component of velocity remains unchanged, therefore the remaining distance (R – x) is covered in the backward direction and projectile falls a distance (R – 2x) ahead of the point O as shown in figure.



Case II : If $x < \frac{R}{2}$

Direction of x component of velocity is reversed but its magnitude remains the same and y component of velocity remains unchanged, therefore the remaining distance (R – x) is covered in the backward direction and projectile falls a distance (R – 2x) behind the the point O as shown in figure.